BERGMAN KERNELS AND EQUILIBRIUM MEASURES FOR AMPLE LINE BUNDLES

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Abstract. Let L be an ample holomorphic line bundle over a compact complex Hermitian manifold X. Any fixed smooth hermitian metric ϕ on L induces a Hilbert space structure on the space of global global holomorphic sections with values in the kth tensor power of L. In this paper various convergence results are obtained for the corresponding Bergman kernels. The convergence is studied in the large k limit and is expressed in terms of the equilibrium metric associated to the fixed metric ϕ, as well as in terms of the Monge-Ampere measure of the metric ϕ itself on a certain support set. It is also shown that the equilibrium metric has Lipschitz continuous first derivatives. These results can be seen as generalizations of well-known results concerning the case when the curvature of the fixed metric ϕ is positive (the corresponding equilibrium metric is then simply ϕ itself).

Contents

1. Introduction

Let L be an ample holomorphic line bundle over a compact complex manifold X of dimension n. Fix an Hermitian fiber metric, denoted by ϕ, on L which is smooth and a smooth volume form on ω_n on X. The curvature form of the metric ϕ may be written as $dd^c\phi$ (see section 1.3 for definitions and further notation). Denote by $\mathcal{H}(X, L^k)$ the Hilbert space obtained by equipping the space $H^0(X, L^k)$ of global holomorphic sections with values in a tensor power L^k with the norm induced by the given smooth metric ϕ on L and the volume form ω_n. The *Bergman kernel* of the Hilbert space $\mathcal{H}(X, L^k)$ is the integral kernel of the orthogonal projection from the space of all smooth sections with values in L^k onto $\mathcal{H}(X, L^k)$. It may be represented by a holomorphic section $K_k(x, y)$ of the pulled back line bundle $L^k \boxtimes \overline{L}^k$ over $X \times \overline{X}$ (formula 3.1).

In the case when the curvature form $dd^c\phi$ is *positive,* the asymptotic properties of the Bergman kernel $K_k(x, y)$ as k tends to infinity has been

studied thoroughly with numerous applications in complex geometry and mathematical physics. For example, $K_k(x,y)$ admits a complete local asymptotic expansion in powers of k; the Tian-Zelditch-Catlin expansion (see [25, 5] and references therein). The point is that when the curvature form $dd^c\phi$ is globally positive, the Bergman kernel asymptotics at a fixed point may be localized and hence only depend (up to negligable terms) on the covariant derivatives of $dd^c\phi$ at the fixed point.

The aim of the present paper is to study the case of a general smooth metric ϕ on L, where global effects become important and where there appears to be very few previous general results. We will consider three natural *positive* measures on X associated to the setup introduced above. First the *equilibrium measure*

$$(dd^c\phi_e)^n/n!,$$

where ϕ_e is the *equilibrium metric* defined by the upper envelope 2.1, and then the large k limit of the measures

$$k^{-n}B_k\omega_n, \tag{1.1}$$

where $B_k(x) := K_k(x,x)e^{-k\phi}$ will be referred to as the *Bergman function* and of the measure

$$(dd^c(k^{-1}\ln K_k(x,x)))^n/n!,$$

often referred to as the kth *Bergman volume form* on X associated to (L,ϕ).

It is not hard to see that the total integrals of all three measures coincide (and equal the total integral over X of the (possibly non-positive) form $(dd^c\phi)^n$). The main point of the present paper is to show the corresponding *local* statement. In fact, all three measures will be shown to coincide with the measure

$$1_D(dd^c\phi)^n/n!$$

where 1_D is the characteristic function the D in X where $\phi_e = \phi$ (corollary 3.4, theorem 3.3 and 3.6). In the case when the metric ϕ has a semi-positive curvature form, $\phi_e = \phi$, i.e. the set D equals all of X. The main results may be suggestively formulated in the following form:

$$K_k(x,x) := (k^n\det(dd^c\phi_e)(x) + ...)e^{k\phi_e(x,x)},$$

where the dots indicate terms of lower order in k. On the set D, the equilibrium metric ϕ_e may be replaced by ϕ and away from the set D the Bergman function $K_k(x,x)e^{-k\phi(x,x)}$ becomes exponentially small in k. Moreover, for any interior point of D where the curvature form of ϕ is positive we will show (theorem 3.8) that the Bergman kernel $K_k(x,y)$, i.e. with two arguments, admits a complete local asymptotic expansion in powers of k, such that the coefficients of the corresponding symbol expansion coincide with the Tian-Zelditch-Catlin expansion for a *positive* Hermitian holomorphic line bundle. Moreover, it will be shown (theorem

3.7) that globally on $X \times X$ the following asymptotics hold for the pointwise norm $|K_k(x,y)|_{k\phi}$:

$$k^{-n} |K_k(x,y)|^2_{k\phi} \, \omega_n(x) \wedge \omega_n(y) \to \Delta \wedge 1_{D \cap X(0)} (dd^c\phi)^n/n! \ ,$$

weakly as measures on $X \times X$, where Δ is the current of integration along the diagonal in $X \times X$. A crucial step in the present approach is to first show the $\mathcal{C}^{1,1}$-regularity of the equilibrium metric ϕ_e (theorem 2.3), which is of independent interest.

Finally, the setup above will be adapted to the case when the Hilbert space $\mathcal{H}(X, L^k)$ is replaced by the subspace of all sections vanishing to high order along a fixed divisor in X. The point is that even in the case when the curvature form of the metric ϕ is positive, the introduction of the divisor is essentially equivalent to studying a (singular) metric with *negative* curvature concentrated along the divisor.

1.1. **Comparison with previous results.** The present paper can be seen as a global geometric version of the situation recently studied in [4], where the role of the Hilbert space $\mathcal{H}(X, L^k)$ was played by the space of all polynomials in $\mathbb{C}^n$ of total degree less than k, equipped with a weighted norm. In fact, apart from the $\mathcal{C}^{1,1}$-regularity of the equilibrium metric ϕ_e, the adaptation to the present setting is fairly straight forward. The proof of the $\mathcal{C}^{1,1}$-regularity is partly modeled on the proof of Bedford-Taylor [1, 20] for $\mathcal{C}^{1,1}-$regularity of the solution of the Dirichlet problem (with smooth boundary data) for the complex Monge-Ampere equation in the unit-ball in $\mathbb{C}^n$. The result should also be compared to various $\mathcal{C}^{1,1}-$results for boundary value problems for complex Monge-Ampere equations on manifolds with boundary [8, 9], intimately related to the study of the geometry of the space of Kähler metrics on a Kähler manifold (see also [21, 6] for other relations to Bergman kernels in the latter context). However, the present situation rather corresponds to a *free* boundary value problem (compare remark 2.4).

In the case of sections vanishing along a fixed divisor (section 4) and under the further assumtion that the curvature form $dd^c\phi$ is positive, similar results have independently been obtained by Julien Keller, Gabor Szekelyhidi and Richard Thomas [17], but with a different algebro-geometric characerization of the set D_Z in formula 4.1.

As in the case studied in [3] (part 1), where the curvature form of the metric ϕ was assumed to be semi-positive the present approach to the Bergman kernel asymptotics is based on the use of "local holomorphic Morse-inequalities", which are local version of the global ones introduced by Demailly [10]. These inequalities are then combined with some global pluripotential theory, based on the recent work [15] by Guedj-Zeriahi. Further references and comments on the relation to the study of random polynomials (and holomorphic sections), random eigenvalues of normal matrices and various diffusion-controlled growth processes studied in the physics literature can be found in [4].

1.2. **Further generalizations.** It can be further shown that the main results in this paper may be generalized to *any* line bundle L over a Kähler manifold. The results are then closely related to the study of the volume of a line bundle over a Kähler manifold [7]. The main general case is when the bundle L is big (i.e. the dimension of $H^0(X, L^k)$ is of the order k^n). Then all the results obtained in the ample case still hold if the Monge-Ampere measure $(dd^c\phi)^n$ is replaced by $1_{X-F}(dd^c(1_{X-F}\phi))^n$, where F is a certain analytic variety in X, naturally associated to L. The main (technical) point is that the proof of the $\mathcal{C}^{1,1}-$ regularity of ϕ_e goes through on the complement of F. Finally, in the case when L is not big it can be shown that the convergence results of the Bergman kernels simply say that $0 = 0$. The details will appear elsewhere.

Acknowledgement 1.1. It is a pleasure to thank Jean-Pierre Demailly and Sebastian Boucksom for several illuminating discussions on the topic of the present paper. In particular, the approach to the proof of the regularity theorem 2.3 was suggested by Jean-Pierre Demailly. Also thanks to Julien Keller for informing me about the work [17].

1.3. **General notation**[1]**.** Let (L, ϕ) be an Hermitian holomorphic line bundle over a compact complex manifold X. The fixed Hermitian fiber metric on L will be denoted by ϕ. In practice, ϕ is considered as a collection of local *smooth* functions. Namely, let s^U be a local holomorphic trivializing section of L over an open set U then locally, $\left|s^U(z)\right|^2_\phi =: e^{-\phi^U(z)}$, where ϕ^U is in the class $\mathcal{C}^2$, i.e. it has continuous derivatives of order two. If α_k is a holomorphic section with values in L^k, then over U it may be locally written as $\alpha_k = f_k^U \cdot (s^U)^{\otimes k}$, where f_k^U is a local holomorphic function. In order to simplify the notation we will usually omit the dependence on the set U. The point-wise norm of α_k may then be locally expressed as

$$|\alpha_k|^2_{k\phi} = |f_k|^2 e^{-k\phi}. \tag{1.2}$$

The canonical curvature two-form of L is the global form on X, locally expressed as $\partial\overline{\partial}\phi$ and the normalized curvature form $i\partial\overline{\partial}\phi/2\pi = dd^c\phi$ (where $d^c := i(-\partial + \overline{\partial})/4\pi$) represents the first Chern class $c_1(L)$ of L in the second real de Rham cohomology group of X. The curvature form of a smooth metric is said to be *positive* at the point x if the local Hermitian matrix $(\frac{\partial^2\phi}{\partial z_i \partial \bar{z_j}})$ is positive definite at the point x (i.e. $dd^c\phi_x > 0$). This means that the curvature is positive when $\phi(z)$ is strictly *plurisubharmonic* i.e. strictly subharmonic along complex lines. We let

$$X(0) := \{x \in X : dd^c\phi_x > 0\}$$

A line bundle L is *ample* if admits some smooth metric whose curvature is positive on all of X.

More generally, a metric ϕ' on L is called (possibly) *singular* if $|\phi'|$ is locally integrable. Then the curvature is well-defined as a $(1,1)-$current

[1] general references for this section are the books [14, 11].

on X. The curvature current of a singular metric is called *positive* if ϕ' may be locally represented by a plurisubharmonic function (in particular, ϕ' takes values in $[-\infty,\infty[$ and is upper semi-continuous (u.s.c)). In particular , any section α_k as above induces such a singular metric on L, locally represented by $\phi' = \frac{1}{k}\ln|f_k|^2$. If Y is a complex manifold we will denote by $PSH(Y)$ and $SPSH(Y)$ the space of all plurisubharmonic and strictly plurisubharmonic functions, respectively.

Fixing an Hermitian metric two-form ω on X (with associated volume form ω_n) the Hilbert space $\mathcal{H}(X,L^k)$ is defined as the space $H^0(X,L^k)$ with the norm

$$\|\alpha_k\|^2_{k\phi} (= \int_X |f_k|^2 e^{-k\phi(z)}\omega_n), \tag{1.3}$$

using a suggestive notation in the last equality (compare formula 1.2).

2. Equilibrium measures for line bundles

Let L be a line bundle over a compact complex manifold X. Given a smooth metric ϕ on L the corresponding "equilibrium metric" ϕ_e is defined as the envelope

$$\phi_e(x) = \sup\left\{\widetilde{\phi}(x) : \widetilde{\phi} \in \mathcal{L}_{(X,L)},\ \widetilde{\phi} \leq \phi \text{ on } X\right\}. \tag{2.1}$$

where $\mathcal{L}_{(X,L)}$ is the class consisting of all (possibly singular) metrics on L with positive curvature form. If ϕ_e is not u.s.c it should be replace it with its u.s.c regularization. Then ϕ_e is also in the class $\mathcal{L}_{(X,L)}$ [15]. In the case when ϕ_e is locally bounded, the corresponding *equilibrium measure* is defined as the Monge-Ampere measure $(dd^c\phi_e)^n/n!$ (see [15] for the definition of the Monge-Ampere measure of a locally bounded metric, based on the work [1] in $\mathbb{C}^n$). When L is ample ϕ_e is clearly locally bounded and it can be shown directly, by adapting the corresponding proof in $\mathbb{C}^n$ (see the appendix in [22]) that $(dd^c\phi_e)^n/n!$ vanishes on the complement of the set.

$$D := \{\phi_e = \phi\} \subset X \tag{2.2}$$

However, the previous vanishing will also be a corollary of theorem 3.3 below.

2.1. $\mathcal{C}^{1,1}-$**regularity for ample line bundles.** In this section we will prove that the equilibrium metric ϕ_e associated to a smooth metric on an *ample* line bundle L is locally in the class $\mathcal{C}^{1,1}$. As in [4], where the manifold X was taken as $\mathbb{C}^n$, the proof is modeled on the proof of Bedford-Taylor [1, 20, 12] for $\mathcal{C}^{1,1}-$regularity of the solution of the Dirichlet problem (with smooth boundary data) for the complex Monge-Ampere equation in the unit-ball in $\mathbb{C}^n$. However, as opposed to $\mathbb{C}^n$ and the unit-ball a generic compact Kähler manifold X has no global holomorphic vector fields. In order to circumvent this difficulty we will reduce the regularity

problem on X to a problem on the pseudoconvex manifold Y, where Y is the total space of the dual line bundle L^*, identifying the base X with its embedding as the zero-section in Y. To any given (possibly singular) metric ϕ on L we may associate the "squared norm function" h_ϕ on Y, where locally

$$h_\phi(z,w) = |w|^2 \exp(\phi(z)).$$

In this way we obtain a bijection

$$\mathcal{L}_{(X,L)} \leftrightarrow \mathcal{L}_Y, \ \ \phi \mapsto h_\phi \tag{2.3}$$

where $\mathcal{L}_Y$ is the class of all positively 2-homogeneous plurisubharmonic functions on Y :

$$\mathcal{L}_Y := \{h \in PSH(Y) : \ h(\lambda\cdot) = |\lambda|^2 h(\cdot)\}, \tag{2.4}$$

using the natural multiplicative action of $\mathbb{C}^*$ on the fibers of Y over X. Now we define

$$h_e := \sup \{h \in \mathcal{L}_Y : \ h \leq h_\phi \text{ on } X\}). \tag{2.5}$$

Then clearly, h_e corresponds to the equilibrium metric ϕ_e under the bijection 2.3. The following lemma will allow us to "homogenize" plurisubharmonic functions.

Lemma 2.1. *Suppose that the function f is spsh, $f = 0$ on X, and f is S^1-invariant in a neighbourhood of the sublevel set $f^{-1}[-\infty, c]$, $c > 0$. Then there is a function $\widetilde{f}$ in the class $\mathcal{L}_Y$ such that $\widetilde{f} = f$ on the level set $f^{-1}(c)$.*

Proof. First observe that $f^{-1}(c)$ is an S^1-bundle subbundle of Y over X. Indeed, since f is assumed to be a spsh function on $f^{-1}[-\infty, c]$ it follows, from the maximum principle applied to discs in each fiber, that f is strictly increasing along the fibers of Y over X. Hence, since $f = 0$ on the zero-section and $c > 0$ any y in $Y - X$ may be written in a unique way as $y = r\sigma$, where $f(\sigma) = c$ and $r > 0$ and we may define $\widetilde{f}$ by

$$\widetilde{f}(r\sigma) := r^2 \widetilde{f}(\sigma).$$

Finally, to see that $\widetilde{f}$ is psh note that since f is strictly increasing along the fibers we have that $\widetilde{f} - c$ is a defining function for the domain $f^{-1}[-\infty, c]$, which is pseudoconvex, since f is psh. In the case when f is smooth and c is a regular value it follows that $(dd^c \widetilde{f}) \geq 0$ along the holomorphic subbundle $T^{1,0}(f^{-1}[-\infty, c])$.By homogeneity this means that $\widetilde{f}$ is psh on all of Y. Finally, the general case may be obtained by a local approximation argument. □

The next (essentially well-known) lemma provides the vector fields needed in the approach of Bedford-Taylor:

Lemma 2.2. *Assume that the line bundle L is ample. For any given point y_0 in $Y - X$ there are global holomorphic vector fields $V_1, ...V_{n+1}$ (i.e. elements of $H^0(Y, TY)$) such that their restriction to y_0 span the tangent space TY_{y_0} and such that V_i vanish on X.*

Proof. It is well-known [11] that on a Stein-space M any holomorphic coherent sheaf is globally generated (i.e. it has the spanning property stated in the lemma). Hence the lemma could be obtained by observing that Y may be blown-down to a Stein space after contracting X to a point so that the push-forward of TY becomes a coherent sheaf. But for completeness we give a somewhat more explicit argument. First note that Y may be compactified by the following fiber-wise projectivized vector bundle:

$$\widehat{Y} := \mathbb{P}(L^* \oplus \underline{\mathbb{C}}),$$

where $\underline{\mathbb{C}}$ denotes the trivial line bundle over X. Denote by $\mathcal{O}(1)$ the line bundle over $\widehat{Y}$ whose restriction to each fiber (i.e. a one-dimensional complex space $\mathbb{P}^1$) is the induced hyperplane line bundle. Next, observe that the line bundle

$$\widehat{L} := (\pi^*(L) \otimes \mathcal{O}(1))$$

over $\widehat{Y}$ is ample if L is, where π denotes the natural projection from $\widehat{Y}$ to X. Indeed, any given smooth metric ϕ_+ on L with positive curvature induces a metric $\widehat{\phi}$ on $\widehat{L}$ with positive curvature, expressed as

$$\widehat{\phi} = \pi^*\phi_+ + \ln(1 + h_{\phi_+})$$

on Y (extending to $\widehat{Y}$). Now it is well-known (for example using Hörmander's L^2-estimates [11]), that for any ample line bundle $\widehat{L}$ and holomorphic vector bundle E on a compact manifold $\widehat{Y}$ the bundle $E \otimes \widehat{L}^{k_0}$is globally generated for k_0 sufficiently large. Setting $E = T\widehat{Y}$ and restricting to Y in $\widehat{Y}$ shows that $TY \otimes \pi^*(L)^{k_0}$ is globally generated on Y (since $\mathcal{O}(1)$ is trivial on Y). Finally, observe that

$$\pi^*(L) = (\pi^*(L^*))^{-1} = [X]^{-1},$$

where $[X]$ is the divisor in Y determined by the embedding of X as the base. Indeed, X is embedded as the zero-set of the tautological section of $\pi^*(L^*)$ over $Y (= L^*)$. Hence, the sections of $TY \otimes \pi^*(L)^{k_0}$ may be identified with sections in TY vanishing to order k_0 on X. This finishes the proof of the lemma. □

For any given smooth vector field V on Y and compact subset K of Y, we denote by $exp(tV)$ the corresponding flow which is well-defined for any "time" t in $[0, t_K]$, i.e. the family of smooth maps indexed by t such that

$$\frac{d}{dt} f(exp(tV)(y)) = df[V]_{exp(tV)(y)} \tag{2.6}$$

for any smooth function f and point y on Y. We will also use the notation $exp(V) := exp(1V)$. In the following we will just be interested in an arbitrarily small neighbourhood of X in Y and we will tacitly take t sufficiently small in order that the flow exists (or equivalently, rescale V).

Now fix a point y_0 in $Y-X$. Combining the previous lemma with the inverse function theorem gives local "exponential" holomorphic coordinates centered at y_0, i.e a local biholomorphism

$$\mathbb{C}^{n+1} \to U(y), \ \ \lambda \mapsto exp((V(\lambda)(y_0), \ \ V(\lambda) := \sum \lambda_i V_i)$$

Using that the vector fields V_i necessarily also span TY_{y_1} for y_1 close to y_0 it can be checked that in order to prove that a function f is locally Lipschitz continuous on a compact subset of Y it is enough to, for each fixed point y_0, prove an estimate of the form

$$|f(exp(V(\lambda))(y_0)) - f(y_0)| \leq C\,|\lambda| \tag{2.7}$$

for some constant C only depending on the function f.

Theorem 2.3. *Suppose that L is an ample line bundle and that the given metric ϕ on L is smooth (i.e. in the class $\mathcal{C}^2$). Then*

(a) ϕ_e is locally in the class $\mathcal{C}^{1,1}$, i.e. ϕ_e is differentiable and all of its first partial derivatives are locally Lipschitz continuous.

(b) The Monge-Ampere measure of ϕ_e is absolutely continuous with respect to any given volume form and coincides with the corresponding L^∞_{loc} $(n,n)-$form obtained by a point-wise calculation:

$$(dd^c\phi_e)^n = \det(dd^c\phi_e)\omega_n \tag{2.8}$$

(c) the following identity holds almost everywhere on the set $D = \{\phi_e = \phi\}$:

$$\det(dd^c\phi_e) = \det(dd^c\phi) \tag{2.9}$$

Proof. To prove (a) it is, by the bijection 2.3, equivalent to prove that h_e (defined by 2.5) is locally $\mathcal{C}^{1,1}$ on $Y-X$. Moreover, by homogeneity it is enough to show that there is a neighbourhood U of X in Y such that h_e is locally $\mathcal{C}^{1,1}$ on $U-X$

Step1: h_e is locally Lipschitz continuous on $Y-X$.

To see this fix a point y_0 in $Y-X$. For simplicity we first assume that h_e is *strictly* plurisubharmonic on $Y-X$ (the assumption will be removed in the end of the argument). Let

$$g(y) := h_e(exp(V(\lambda))(y)), \ \ \widehat{g}(y) := \text{u.s.c}(\sup_{\theta\in[0,2\pi} g(e^{i\theta}y))$$

using the natural multiplicative action of $\mathbb{C}^*$ on the fibers of Y over X and where u.s.c. denotes the upper-semicontinous regularization. Then both g and $\widehat{g}$ are psh functions (using that the family $g(e^{i\theta}\cdot)$ of psh functions is locally bounded). Note that

$$h_e(exp(V(\lambda))(y_0) =: g(y_0) \leq \widehat{g}(y_0) = \widetilde{\widehat{g}}(y_0),$$

where $\widetilde{\widehat{g}}$ is the function in the class $\mathcal{L}_Y$ obtained from lemma 2.1 applied to $f = \widehat{g}$ and $c = \widehat{g}(y_0)$ (note that $\widehat{g}$ is still *strictly* plurisubharmonic,

since the flow $exp(V(\lambda))$ fixes X). Moreover, since by definition $h_e \leq h_\phi$ we have the following bound on the levelset $\widetilde{\widetilde{g}}^{-1}(c)$:

$$\widetilde{\widetilde{g}}((y) \leq \sup_{\theta\in[0,2\pi]} h_\phi((exp(V(\lambda))(e^{i\theta}y) \leq \sup_{\theta\in[0,2\pi]} h_\phi(e^{i\theta}y) + C\,|\lambda|\,, \tag{2.10}$$

using that h_ϕ is locally Lipschitz in the last inequality. Indeed, since h_ϕ is in the class $\mathcal{C}^1$ the property 2.6 of the flow gives

$$h_\phi((exp(V(\lambda))(\zeta) - h_\phi(\zeta) = \int_0^1 dh_\phi[V(\lambda)]_{exp(tV(\lambda)(y)}dt$$

Hence, since $V(\lambda) := |\lambda|\,(\sum \frac{\lambda_i}{|\lambda|}V_i)$, the constant C in 2.10 may be taken to be

$$C = \sup_{y\in K, i=1,..N+1} |dh_\phi[V_i]_y|$$

for some compact neighbourhood K of X in Y. Since h_ϕ is S^1-invariant and $\widetilde{\widetilde{g}}^{-1}(c)$ is compact, 2.10 gives that

$$\widetilde{\widetilde{g}} - C\,|\lambda| \leq h_\phi \tag{2.11}$$

on $\widetilde{\widetilde{g}}^{-1}(c)$ and hence, by homogeneity, on all of Y. This shows that the function $\widetilde{\widetilde{g}} - C\,|\lambda|$ is a contender for the supremum in the definition 2.5 of h_e and hence bounded by h_e. All in all we get that

$$h_e(exp(V(\lambda))(y_0) \leq \widetilde{\widetilde{g}}(y_0) \leq h_\phi(y_0) + C\,|\lambda|\,.$$

The other side of the inequality 2.7 for $f = h_e$ is obtained after replacing λ by $-\lambda$. Finally, in order to remove the simplifying assumption that h_e be *strictly* plurisubharmonic on $Y - X$ we apply the previous argument to get the same bounds on

$$h_\delta := h_e(1-\delta) + \delta h_+$$

where h_+ is spsh and in the class $\mathcal{L}_Y$ (if L is ample than h_+ clearly exists). Finally, letting δ tend to zero, finishes the proof of Step 1.

Step2: dh_e exists and is locally Lipschitz continuous on $Y - X$.

Following the exposition in [12] it is enough to prove the following inequality:

$$h_e(exp(V(\lambda))(y_0) + h_e(exp(V(-\lambda))(y_0) - 2h_e(y_0) \leq C\,|\lambda|^2\,, \tag{2.12}$$

where the constant only depends on the second derivatives of h_ϕ on some compact subset of Y. Indeed, given this inequality (combined with the fact that h_e is psh) a Taylor expansion of degree 2 gives the following bound close to y_0 for a smooth approximation h_ϵ of h_e :

$$\left|D^2 h_\epsilon\right| \leq C$$

where $h_\epsilon := h_e * \chi_\epsilon$, using a a local regularizing kernel χ_ϵ and where $D^2 h_\epsilon$ denotes the real Hessian matrix of h_ϵ. Letting ϵ tend to 0 then proves Step

2. Finally, to see that the inequality 2.12 holds we apply the argument in Step 1 after replacing g by the psh function

$$f(y) := (h_e(exp(V(\lambda))(y) + h_e(exp(V(-\lambda))(y))/2$$

to get

$$f(y) \leq \widetilde{\widehat{f}}((y) \leq \sup_{\theta \in [0,2\pi]} (h_\phi((exp(V(\lambda))(e^{i\theta}y) + (exp(V(-\lambda))(e^{i\theta}y))/2$$

Next, observe that for each fixed θ the function $h_\phi(e^{i\theta}y)$ is in the class $\mathcal{C}^2$. Hence, a Taylor expansion of degree 2 gives

$$\widetilde{\widehat{f}}((y) \leq \sup_{\theta \in [0,2\pi]} ((h_\phi(e^{i\theta}y)) + C\left|\lambda\right|^2) = h_\phi(y) + C\left|\lambda\right|^2)$$

where the constant C may be taken as a constant times $\sup_K \left|D^2 h_\phi\right|$. This shows that $\widetilde{\widehat{f}} - C\left|\lambda\right|^2$ is a contender for the supremum in the definition 2.5 of h_e and hence bounded by h_e. All in all we obtain that

$$f(y_0) \leq h_\phi(y_0) + C\left|\lambda\right|^2,$$

which proves the inequality 2.12, finishing the proof of Step2.

(b) By the $\mathcal{C}^{1,1}-$regularity, the derivatives $\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\phi_e$ are in L^∞_{loc} and it is well-known that this implies the identity 4.5 for the Monge-Ampere measure. Finally, to see that 4.6 holds, it is enough to prove that locally

$$\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}(\phi_e - \phi) = 0$$

almost everywhere on $D = \{\phi_e = \phi\}$. To this end we apply a calculus lemma in [18] (page 53) to the $\mathcal{C}^{1,1}-$function $\phi_e - \phi$ (following the approach in [16]), which even gives the corresponding identity between all real second order partial derivatives almost everywhere on D. □

Remark 2.4. Fix a metric ϕ_+ on L with positive curvature. Then $\omega_+ := dd^c\phi_+$ is a Kähler metric on X and the fixed metric ϕ on L may be written as $\phi = u + \phi_+$, where u is a smooth *function* on X. Now the pair (u_e, M) where $u_e := \phi_e - \phi_+$ and M is the set $X - D$, may be interpreted as a "weak" solution to the following *free* boundary value problem of Monge-Ampere type[2]:

$$\begin{array}{rcll} (dd^c u_e + \omega_+)^n & = & 0 & \text{on } M \\ u_e & = & u & \text{on } \partial M \\ du_e & = & du & \end{array}$$

The point is that, since the equations are overdetermined, the set M is itself part of the solution. In [16] the $\mathcal{C}^{1,1}-$regularity of ϕ_e in the case when $X = \mathbb{C}$ (corresponding to the setup in [22]) was deduced from the regularity of a free boundary value problem.

[2]since there is a priori no control on the regularity of the set M, it does not really make sense to write ∂M and the boundary condition should hence be interpreted in a suitable "weak" sense.

3. Bergman kernel asymptotics

Denote by $\mathcal{H}(X,L^k)$ the Hilbert space obtained by equipping the vector space $H^0(X,L^k)$ with the norm 1.3 induced by the given smooth metric ϕ on L and the volume form ω_n. Let (ψ_i) be an orthonormal base for $\mathcal{H}(X,L^k)$. The *Bergman kernel* of the Hilbert space $\mathcal{H}(X,L^k)$ is the integral kernel of the orthogonal projection from the space of all smooth sections with values in L^k onto $\mathcal{H}(X,L^k)$. It may be represented by the holomorphic section

$$K_k(x,y)=\sum_i \psi_i(x)\otimes\overline{\psi_i(y)}. \tag{3.1}$$

of the pulled back line bundle $L^k\boxtimes\overline{L}^k$ over $X\times\overline{X}$. The restriction of K_k to the diagonal is a section of $L^k\otimes\overline{L}^k$ and we let $B_k(x)=|K_k(x,x)|_{k\phi}$ $(=|K_k(x,x)|\,e^{-k\phi(x)})$ be its point wise norm:

$$B_k(x)=\sum_i |\psi_i(x)|^2_{k\phi}\,. \tag{3.2}$$

We will refer to $B_k(x)$ as the *Bergman function* of $\mathcal{H}(X,L^k)$. It has the following extremal property:

$$B_k(x)=\sup\left\{|\alpha_k(x)|^2_{k\phi}:\ \alpha_k\in\mathcal{H}(X,L^k),\ \|\alpha_k\|^2_{k\phi}\leq 1\right\} \tag{3.3}$$

Moreover, integrating 3.2 shows that B_k is a "dimensional density" of the space $\mathcal{H}(X,L^k)$:

$$\int_X B_k\omega_n=\dim\mathcal{H}(X,L^k) \tag{3.4}$$

Now if L is an ample line bundle, the dimension of $\mathcal{H}(X,L^k)$ is explicitly known to the leading order in k [14] giving

$$\lim_{k\to\infty}\int_X k^{-n}B_k\omega_n=\int_X c_1(L)^n(=\int_X (dd^c\phi_e)^n), \tag{3.5}$$

where we have represented the first Chern class $c_1(L)$ by the *positive* curvature current $dd^c\phi_e$ of the equilibrium metric ϕ_e, using that $(dd^c\phi_e)^n$ is well-defined when L is ample.

The following "local Morse inequality" estimates B_k point-wise from above for a general bundle:

Lemma 3.1. (Local Morse inequalities) *Let ϕ be a smooth metric on a holomorphic line bundle L over a compact manifold X. Then the following upper bound holds on X :*

$$k^{-n}B_k\leq C_k 1_{X(0)}\det(dd^c\phi),$$

where the sequence C_k of positive numbers tends to one and $X(0)$ is the set where $dd^c\phi>0$.

See [2] for the more general corresponding result for $\overline{\partial}-$harmonic $(0,q)$-forms with values in a high power of an Hermitian line bundle. The present case (i.e. $q=0$) is a simple consequence of the mean-value property of holomorphic functions applied to a poly-disc Δ_k of radius $\ln k/\sqrt{k}$ centered at the origin in $\mathbb{C}^n$ (see the proof in [3]). In fact, the proof gives the following stronger local statement:

$$\limsup_k k^{-n} \left|f_k(0)\right|^2 e^{-k\phi(z)} / \left\|f_k\right\|^2_{k\phi,\Delta_k(z)} \leq 1_{X(0)}(0) \det(dd^c\phi), \tag{3.6}$$

where f_k is holomorphic function defined in a fixed neighbourhood of the origin in $\mathbb{C}^n$.

The estimate in the previous lemma can be considerably sharpened on the complement of D (formula 2.2), as shown by the following lemma:

Lemma 3.2. *Let ϕ be a smooth metric on a holomorphic line bundle L over a compact manifold X. Then the following inequality holds on all of X:*

$$B_k k^{-n} \leq C_k e^{-k(\phi-\phi_e)} \tag{3.7}$$

where the sequence C_k of positive numbers tends to $\sup_X \det(dd^c\phi)$. *In particular,*

$$\lim \int_{D^c} k^{-n} B_k \omega_n = 0 \tag{3.8}$$

Proof. By the extremal property 3.3 of B_k it is enough to prove the lemma with $B_k k^{-n}$replaced by $\left|\alpha_k\right|^2_{k\phi}$, locally represented by $\left|f_k\right| e^{-k\phi}$, for any element α_k in $\mathcal{H}(X,L^k)$ with global norm equal to k^{-n}. The Morse inequalities in the previous lemma give that

$$\left|f_k\right|^2 e^{-k\phi} \leq C_k$$

with C_k as in the statement of the present lemma. Equivalently,

$$\frac{1}{k}\ln\left|f_k\right|^2 - \frac{1}{k}C_k \leq \phi$$

Hence, the singular metric on L determined by $\frac{1}{k}\ln\left|f_k\right|^2 - \frac{1}{k}C_k$ is a candidate for the sup in the definition 2.1 of ϕ_e and is hence bounded by ϕ_e. Thus,

$$B_k k^{-n} = \left|f_k\right|^2 e^{-k\phi} \leq C_k e^{k\phi_e} e^{-k\phi}.$$

□

Finally, the vanishing 3.8 follows from the dominated convergence theorem, since the right hand side in the previous inequality tends to zero precisely on the complement of D.

Theorem 3.3. *Let L be an ample line bundle over X and let B_k be the Bergman function of the Hilbert space $\mathcal{H}(X,L^k)$. Then*

$$k^{-n} B_k \to 1_{D\cap X(0)} \det(dd^c\phi), \tag{3.9}$$

in $L^1(X,\omega_n)$, where $X(0)$ is the set where $dd^c\phi > 0$ and D is the set 2.2.

Proof. First observe that, by the exponential decay in lemma 3.2,

$$\lim_{k\to\infty} k^{-n}B_k(x)=0,\ \ x\in D^c$$

Next, observe that it is enough to prove that

$$\lim_{k\to\infty}\int_D k^{-n}B_k\omega_n=\int_{D\cap X(0)}(dd^c\phi)^n/n! \tag{3.10}$$

Indeed, given this equality the local Morse inequalities (lemma 3.1), then force the convergence 4.8 on the compact set D. The proof proceeds precisely as in [3] (part 1, section 2).

Finally, to prove that 3.10 does hold, first note that

$$\int_X (dd^c\phi_e)^n=\lim_{k\to\infty}\int_D k^{-n}B_k\omega_n\leq\int_{D\cap X(0)}(dd^c\phi)^n/n! \tag{3.11}$$

where we have combined formula 3.5 and 3.8 to get the equality and then used the local Morse inequalities (lemma 3.1) in the inequality (also using the dominated convergence theorem). Finally, by formula 4.6 in theorem 2.3 we may replace ϕ with ϕ_e in the right hand side, showing the the right hand side in the previous equality is equal to $\int_{D\cap X(0)}(dd^c\phi_e)^n$. But since $(dd^c\phi_e)^n$ is a positive measure this can only happen if all inequalities in 3.11 are actually equalities, which proves 3.10 and finishes the proof of the theorem. □

Combining the previous theorem with the regularity theorem 2.3 now gives the following

Corollary 3.4. *The equilibrium measure corresponding to the smooth metric ϕ on L is given by*

$$(dd^c\phi_e)^n/n!=1_{D\cap X(0)}(dd^c\phi)^n/n!,$$

where $D=\{\phi_e=\phi\}$.

3.1. **The Bergman metric.** The Hilbert space $\mathcal{H}(X,L^k)$ induces a metric on the line bundle L in the class $\mathcal{L}_{(X,L)}$ which may be expressed as

$$k^{-1}\ln K_k(x,x),$$

often referred to as the kth Bergman metric on L. If L is an ample line bundle, then this is the smooth metric on L obtained as the pull-back of the Fubini-Study metric on the hyperplane line bundle $\mathcal{O}(1)$ over $\mathbb{P}^N(=\mathbb{P}\mathcal{H}(X,L^k))$ (compare example 5.1 in section 5) under the Kodaira map

$$X\to\mathbb{P}\mathcal{H}(X,L^k),\ \ y\mapsto(\Psi_1(x):\Psi_2(x)...:\Psi_N(x))\ ,$$

for k sufficiently large, where (Ψ_i) is an orthonormal base for $\mathcal{H}(X,L^k)$ [14].

We will make use of the following well-known extension lemma, which follows from the Ohsawa-Takegoshi theorem (compare [6]):

Lemma 3.5. *Let F be a line bundle with a (possibly singular) metric ϕ_L such that the curvature $dd^c\phi_F$ is positive in the sense of currents and let A be an ample line bundle. Then, after possibly replacing A by a sufficiently large tensor power, the following holds: for any point x in X where $\phi \neq \infty$, there is an element α in $H^0(X, F \otimes A)$ such that*

$$|\alpha(x)|_{\phi_F+\phi_A} = 1, \ \ \|\alpha\|_{X,\phi_F+\phi_A} \leq C. \tag{3.12}$$

The constant C is independent of the line bundle F and the point x and depends only on a fixed smooth metric ϕ_A on A.

Now we can prove the following theorem:

Theorem 3.6. *Let L be an ample line bundle and let K_k be the Bergman kernel of the Hilbert space $\mathcal{H}(X, L^k)$. Then the following convergence of Bergman metrics holds:*

$$k^{-1}\ln K_k(x,x) \to \phi_e(x) \tag{3.13}$$

uniformly on X (the rate of convergence is of the order $\ln k/k$). In particular, the corresponding "Bergman volume forms" converge to the equilibrium measure:

$$(dd^c(k^{-1}\ln K_k(x,x)))^n \to (dd^c\phi_e)^n \tag{3.14}$$

weakly as measures.

Proof. In the following proof it will be convenient to let C denote a sufficiently large constant (which may hence vary from line to line). First observe that taking the logarithm of the inequality 3.7 in lemma 3.2 immediately gives the upper bound

$$k^{-1}\ln K_k(x,x) \leq \phi_e(x) + C \ln k/k$$

To get a lower bound, fix a point x_0 in X and note that by the extremal property 3.3 it is enough to find a section α_k such that

$$|\alpha_k(x_0)|_{k\phi_e} \geq 1/C, \ \ \|\alpha_k\|_{X,k\phi} \leq C. \tag{3.15}$$

To this end we take the section α_k furnished by lemma the previous lemma applied to $(F, \phi_F) = (L^{k-k_0}, (k-k_0)\phi_e)$ and $A = L^{k_0}$, for k_0 sufficiently large. Then $L^k = F \otimes A$ gets an induced metric

$$\psi_k = (k-k_0)\phi_e + \phi_A \tag{3.16}$$

and

$$|\alpha_k(y)|_{k\phi_e} \geq 1/C, \ \ \|\alpha_k\|_{X,\psi_k} \leq C,$$

where we have fixed a smooth metric ϕ_A on L^{k_0} with positive curvature such that $\phi_A \leq k_0\phi$. Since, by definition $\phi_e \leq \phi$, this proves 3.15 and hence the theorem.

The Monge-Ampere convergence 3.14 now follows from the uniform convergence 3.13 (see [15]). □

3.2. **The full Bergman kernel.** Combining the convergence in theorem 3.3 with the local inequalities 3.6, gives the following convergence for the point-wise norm of the full Bergman kernel $K_k(x,y)$. The proof is completely analogous to the proof of theorem 2.4 in part 1 of [3].

Theorem 3.7. *Let L be an ample line bundle and let K_k be the Bergman kernel of the Hilbert space $\mathcal{H}(X, L^k)$. Then*

$$k^{-n} \left| K_k(x,y) \right|^2_{k\phi} \omega_n(x) \wedge \omega_n(y) \to \Delta \wedge 1_{D \cap X(0)} (dd^c\phi)^n/n! \ ,$$

*as measures on $X \times X$, in the weak *-topology, where Δ is the current of integration along the diagonal in $X \times X$.*

Finally, we will show that around any interior point of the set $D \cap X(0)$ the Bergman kernel $K_k(x,y)$ admits a complete local asymptotic expansion in powers of k, such that the coefficients of the corresponding symbol expansion coincide with the Tian-Zelditch-Catlin expansion (concerning the case when the curvature form of ϕ is positive on all of X; see [5] and the references therein for the precise meaning of the asymptotic expansion). We will use the notation $\phi(x,y)$ for a fixed almost holomorphic-anti-holomorphic extension of a local representation of the metric ϕ from the diagonal Δ in $\mathbb{C}^n \times \mathbb{C}^n$, i.e. an extension such that the anti-holomorphic derivatives in x and the holomorphic derivatives in y vanish to infinite order on Δ.

Theorem 3.8. *Let L be an ample line bundle and let K_k be the Bergman kernel of the Hilbert space $\mathcal{H}(X, L^k)$. Any interior point in $D \cap X(0)$ has a neighbourhood where $K_k(x,y)e^{-k\phi(x)/2}e^{-k\phi(y)/2}$ admits an asymptotic expansion as*

$$k^n(\det(dd^c\phi)(x) + b_1(x,y)k^{-1} + b_2(x,y)k^{-2} + \ldots)e^{k\phi(x,y)}, \tag{3.17}$$

where b_i are global well-defined functions expressed as polynomials in the covariant derivatives of $dd^c\phi$ (and of the curvature of the metric ω) which can be obtained by the recursion given in [5].

Proof. The proof is obtained by adapting the construction in [5], concerning *positive* Hermitian line bundles, to the present situation. The approach in [5] is to first construct a "local asymptotic Bergman kernel" close to any point where ϕ is smooth and $dd^c\phi > 0$. Hence, the local construction applies to the present situation as well. Then the local kernel is shown to differ from the true kernel by a term of order $O(k^{-\infty})$, by solving a $\overline{\partial}$-equation with a good L^2-estimate. This is possible since $dd^c\phi > 1/C$ *globally* in that case. In the present situation we are done if we can solve

$$\overline{\partial} u_k = g_k, \tag{3.18}$$

where g_k is a $\overline{\partial}-$closed $(0,1)-$form with values in L^k, supported on the interior of the bounded set $D \cap X(0)$, with an estimate

$$\left\| u_k \right\|_{k\phi} \leq C \left\| g_k \right\|_{k\phi} \tag{3.19}$$

To this end we apply the L^2-estimates of Hörmander-Kodaira [11] with the metric ψ_k on L^k (formula 3.16) occurring in the proof of theorem 3.6. This gives a solution u_k satisfying the inequality 3.19 with $k\phi$ replaced by the weight function ψ_k. To see that we actually have the estimate 3.19 (i.e. with the weight $k\phi$ itself) we apply the same argument as in the end of the proof of theorem 3.6 to the left hand side in 3.19. Finally, for the right hand side in 3.19, we use that $\phi = \phi_e$ on the bounded set $D \cap X(0)$ where g_k is supported. $\square$

4. Sections vanishing along a divisor

In this section we will show that the setup in the previous sections can be adapted to the case when the Hilbert space $\mathcal{H}(X, L^k)$ is replaced by the subspace of all sections vanishing to order at least k along a fixed divisor (which we for simplicity take to be irreducible):

$$Z = \{s_Z = 0\},$$

assuming that the line bundle $L \otimes [Z]^{-1}$ is ample (without assuming that L is ample). The point is that this amounts essentially to replacing (L, ϕ) with the Hermitian line bundle $(L \otimes [Z]^{-1}, \phi - \ln(|s_Z|^2))$ to which the previous setup essentially applies if one also takes into account the fact the singular metric $\phi - \ln(|s_Z|^2)$ is equal to infinity on Z. Hence, the corresponding curvature current will be *negative* close to Z, even if the curvature form of ϕ is globally positive.

4.1. **Equilibrium metrics with poles along a divisor.** Let $\mathcal{L}_{(X,L);Z}$ be the subclass of $\mathcal{L}_{(X,L)}$ consisting of all metrics $\widetilde{\phi}$ on L such that the Lelong numbers of $\widetilde{\phi}$ are bounded from below by one along the divisor Z :

$$\nu(\widetilde{\phi})_x \geq 1, \text{ when } x \in Z,$$

where

$$\nu(\widetilde{\phi})_x := \lim_{r \to 0_+} \frac{1}{r^{2n}} \int_{|z| \leq r} dd^c\phi \wedge (dd^c |z|^2)^{n-1}/(n-1)!,$$

with respect to any local coordinate z centered at x. Then we define the associated *equilibrium metric with poles along* Z as

$$\phi_{e.Z}(x) = \sup \left\{ \widetilde{\phi}(x) : \widetilde{\phi} \in \mathcal{L}_{(X,L);Z}, \widetilde{\phi} \leq \phi \text{ on } X \right\}$$

and the following set, compactly included in $X - Z$:

$$D_Z := \{\phi_{e,Z} = \phi\} \tag{4.1}$$

Lemma 4.1. *The following decomposition holds:*

$$\phi_{e,Z} = \psi_e + \ln(|s_Z|^2)$$

where

$$\psi_e(x) = \sup \left\{ \widetilde{\psi}(x) : \widetilde{\psi} \in \mathcal{L}_{(X, L \otimes [Z]^{-1})}, \widetilde{\psi} \leq \phi - \ln(|s_Z|^2) \text{ on } X - Z \right\}.$$

Proof. First observe that

$$\nu(\widetilde{\phi})_x \geq 1, \forall x \in Z \Leftrightarrow \exists (C, U_Z) : \widetilde{\phi} \leq \ln(|s_Z|^2) + C \text{ on } U_Z, \tag{4.2}$$

where U_Z is a neighbourhood of Z. Indeed, this is a direct consequence of the following characterization [11] of the Lelong number at 0 of a germ of a psh function in $\mathbb{C}^n$:

$$\nu(\widetilde{\phi})_0 = \sup\left\{\gamma : \widetilde{\phi} \leq \gamma \ln(|z|^2) + C_\gamma \text{ close to } 0\right\} \tag{4.3}$$

Next, observe that

$$\exists (C, U_Z) : \widetilde{\phi} \leq \ln(|s_Z|^2) + C \text{ on } U_Z \Leftrightarrow \widetilde{\psi} := \widetilde{\phi} - \ln(|s_Z|^2) \in \mathcal{L}_{(X, L\otimes[Z]^{-1})}. \tag{4.4}$$

To see this first note that in general $\widetilde{\psi}$ above determines a (possibly singular) metric on $L\otimes[Z]^{-1}$ over $X-Z$ with positive curvature current $dd^c\widetilde{\psi} = dd^c\widetilde{\phi}$. Now the condition that $\widetilde{\phi} \leq \ln(|s_Z|^2)+C$ close to Z means that $\widetilde{\psi}$ is bounded close to Z, which is equivalent to the fact that it extends to an element of $\mathcal{L}_{(X,L\otimes[Z]^{-1})}$. This follows from the corresponding local extension property of plurisubharmonic functions over an analytic variety (or more generally over any pluripolar set) in $\mathbb{C}^n$ [19].

Finally, combining 4.2 and 4.4 gives a bijection between $\mathcal{L}_{(X,L);Z}$ and $\mathcal{L}_{(X,L\otimes[Z]^{-1})}$. Since, $\widetilde{\phi} \leq \phi$ on X if and only if the equality holds on $X-Z$ (using that $\widetilde{\phi}$ is assumed to be u.s.c) this finishes the proof of the lemma. □

Using the decomposition in the previous lemma the following regularity result is obtained (compare theorem 2.3):

Theorem 4.2. *Suppose that $L\otimes[Z]^{-1}$ is an ample line bundle and that the given metric ϕ on L is smooth. Then*

(a) $\phi_{e,Z}$ is locally in the class $\mathcal{C}^{1,1}$ on $X-Z$, i.e. ϕ_e is differentiable and all of its first partial derivatives are locally Lipschitz continuous. In fact,

$$\phi_{e,Z} = \psi_e + \ln(|s_Z|^2),$$

where ψ_e is locally in the class $\mathcal{C}^{1,1}$ on all of X.

(b) The Monge-Ampere measure of ϕ_e on $X-Z$ is absolutely continuous with respect to any given volume form and coincides with the corresponding L^∞_{loc} $(n,n)-$form obtained by a point-wise calculation:

$$(dd^c\phi_{e,Z})^n = \det(dd^c\phi_{e,Z})\omega_n \ \text{ on } X-Z \tag{4.5}$$

(c) the following identity holds almost everywhere on the set $D_Z = \{\phi_{e,Z} = \phi\}$:

$$\det(dd^c\phi) = \det(dd^c\phi_{e,Z}) = \det(dd^c\psi_e) \tag{4.6}$$

Proof. (a) By the previous lemma it is enough to prove that ψ_e is locally in the class $\mathcal{C}^{1,1}$ on X. Since $L\otimes[Z]^{-1}$ is an ample line bundle over X this would be a direct consequence of theorem 2.3 if $\psi := \phi - \ln(|s_Z|^2)$ were in the class $\mathcal{C}^2$ on all of X and not only on $X-Z$. In order to supply

the necessary modifications of the argument note that, as a locally psh function, ψ_e is necessarily bounded from above close to Z and hence the bound corresponding to the bound 2.11 trivially holds over a neighbourhood of Z, since ψ is equal to infinity along Z. This proves that ψ_e is locally Lipschitz and the rest of the proof of (a) can be modified an a similar way. The proof of (b) and (c) proceeds exactly as in the proof of theorem 2.3. $\square$

4.2. **Bergman kernels vanishing along a divisor.** Now consider the sub Hilbert space $\mathcal{H}_{k,Z}$ of $\mathcal{H}(X, L^k)$, consisting of all sections α_k such that the order of vanishing of α_k along the divisor Z is at least k. Since the later condition means that

$$f_k = g_k \otimes s_Z^{\otimes k}, \;\; g_k \in H^0(X, L^k \otimes [Z]^{-1})$$

the Hilbert space $\mathcal{H}_{k,Z}$ is isomorphic to the vector space $H^0(X, L^k \otimes [Z]^{-1})$ equipped with the norm 1.3 induced by ϕ, under the natural embedding

$$(\cdot) \otimes s_Z^{\otimes k} : \; H^0(X, L^k \otimes [Z]^{-1}) \to H^0(X, L^k) \tag{4.7}$$

We denote by $K_{k,Z}$ and $B_{k,Z}$ the corresponding Bergman kernels and Bergman functions, respectively.

Theorem 4.3. *Assume that the line bundle $L \otimes [Z]^{-1}$ over X is ample. Let $B_{k,Z}$ be the Bergman function of the Hilbert space $\mathcal{H}_{k,Z}$ of all sections vanishing along Z to order at least k. Then*

$$k^{-n} B_{k,Z} \to 1_{D_{e,Z} \cap X(0)} \det(dd^c\phi), \tag{4.8}$$

in $L^1(X, \omega_n)$.

Proof. First note that the local Morse inequalities 3.1 still hold when B_k is replaced by $B_{k,Z}$, since $B_{k,Z} \leq B_k$. Moreover, if f_k is a local representation of en element of $\mathcal{H}_{k,Z}$, the embedding 4.7 and the characterization 4.3 of the Lelong numbers give that $\frac{1}{k}\ln|f_k(z)|^2$(and hence also $k^{-1}\ln K_{k,Z}$) belongs to the class $\mathcal{L}_{(X,L);Z}$. Hence, the proof of lemma 3.7 goes through in the present setting, showing that $k^{-n}B_{k,Z}$ converges (exponentially) to zero on $X - D_Z$. Thus, we get as in the proof of theorem 3.3

$$\lim_k k^{-n} \dim \mathcal{H}_{k,Z} = \lim_k \int_{D_Z} k^{-n} B_{k,Z}\omega_n \leq \int_{D_Z \cap X(0)} (dd^c\phi)^n/n! \tag{4.9}$$

By theorem 4.2 the right hand side is bounded by $\int_{X-Z}(dd^c\psi_e)^n/n!$ which is equal to the top intersection number of $c_1(L \otimes [Z]^{-1})$. Now, since $L \otimes [Z]^{-1}$ is ample this number is equal to the limit of $k^{-n} \dim H^0(X, L^k \otimes [Z]^{-1})$, which in turn equals the left hand side in 4.9. Hence, the inequality in 4.9 is actually an equality. The rest of the proof proceeds word for word as in the proof of theorem 3.3. $\square$

Next, we have the following generalization of theorem 3.6, which can be seen as a global version of the general approximation results for

$(1,1)-$currents of Demailly [13], in the particular case when the current (here given by $dd^c\phi_{e,Z}$) is singular along a divisor.

Theorem 4.4. *Assume that the line bundle $L \otimes [Z]^{-1}$ over X is ample. Let $K_{k,Z}$ be the Bergman kernel of the Hilbert space $\mathcal{H}_{k,Z}$ of all sections vanishing along Z to order at least k. Then*

$$k^{-1}\ln K_{k,Z}(x,x) \to \phi_{e,Z}(x)$$

uniformly on $X - Z$ (the rate of convergence is of the order $\ln k/k$). In particular, the corresponding convergence of the Lelong numbers along Z holds:

$$\nu(k^{-1}\ln K_{k,Z})_x \to \nu(\phi_{e,Z})_x(=1)$$

and the corresponding "Bergman volume forms" converge on $X - Z$ to the equilibrium measure:

$$(dd^c(k^{-1}\ln K_{k,Z}(x,x)))^n \to (dd^c\phi_{e,Z})^n$$

weakly as measures on $X - Z$.

Proof. As above the proof can be reduced to the proof of the corresponding theorem in the case when there is no divisor Z (theorem 3.6). This gives the uniform convergence

$$k^{-1}\ln K_{k,Z}(x,x) - \ln(|s_Z|^2(x)) \to \phi_e(x) - \ln(|s_Z|^2(x))$$

on all of X. The convergence of the Lelong numbers then follows from the characterization 4.3. $\square$

Finally, note that the theorems 3.7 and 3.8 generalize in the corresponding way to the present situation. The explicit statements are omitted.

5. Examples

Finally, we illustrate some of the previous results with the following examples, which can be seen as variants of the setting considered in [4] (compare remark 5.3 below).

Example 5.1. Let X be the $n-$dimensional projective space $\mathbb{P}^n$ and let L be the hyperplane line bundle $\mathcal{O}(1)$. Then $H^0(X, L^k)$ is the space of homogeneous polynomials in $n+1$ homogeneous coordinates $Z_0, Z_1, ..Z_n$. The Fubini-Study metric ϕ_{FS} on $\mathcal{O}(1)$ may be suggestively written as $\phi_{FS}(Z) = \ln(|Z|^2)$ and the Fubini-Study metric ω_{FS} on $\mathbb{P}^n$ is the normalized curvature form $dd^c\phi_{FS}$. Hence the induced norm on $H^0(X, L^k)$ is invariant under the standard action of $SU(n+1)$ on $\mathbb{P}^n$. We may identify $\mathbb{C}^n$ with the "affine piece" $\mathbb{P}^n - H_\infty$ where H_∞ is the "hyperplane at infinity" in $\mathbb{C}^n$ (defined as the set where $Z_0 = 0$). In terms of the standard trivialization of $\mathcal{O}(1)$ over $\mathbb{C}^n$ (obtained by setting $Z_0 = 1$) the space $H^0(Y, L^k)$ may be identified with the space of polynomials $f_k(\zeta)$ in $\mathbb{C}^n_\zeta$ of

total degree at most k and the metric ϕ_{FS} on $\mathcal{O}(1)$ may be represented by the function

$$\phi_{FS}(\zeta) = \ln(1 + |\zeta|^2).$$

Moreover, any smooth metric on $\mathcal{O}(1)$ may be represented by a function $\phi(\zeta)$ satisfying the following necessary growth condition[3]

$$-C + \ln(1 + |\zeta|^2) \leq \phi(\zeta) \leq \ln(1 + |\zeta|^2) + C, \tag{5.1}$$

which makes sure that the norm 1.3, expressed as

$$\|f_k\|^2_{k\phi} := \int_{\mathbb{C}^n} |f_k(\zeta)|^2 \, e^{-k\phi(\zeta)} \omega^n_{FS}/n!$$

is finite precisely when f_k corresponds to a section of $\mathcal{O}(m)$, for $m = 1$. In particular, any smooth compactly supported function $\chi(\zeta)$ determines a *smooth* perturbation

$$\phi_\chi(\zeta) := \phi_{FS}(\zeta) + \chi(\zeta) \tag{5.2}$$

of ϕ_{FS} on $\mathcal{O}(1)$ over $\mathbb{P}^n$, to which the results in section 2 and 3 apply. For example, if χ is a radial function then it can be checked that the graph of the equilibrium metric determined by ϕ_χ is simply the convex hull of the graph of ϕ_χ considered as a function of $v := \ln |\zeta|^2$.

The next example introduces a divisor into the picture, as in section 4.

Example 5.2. Let $X = \mathbb{P}^n$, $L = \mathcal{O}(2)$ and denote by Z the hyperplane at infinity in $\mathbb{C}^n$. Then $L \otimes [Z]^{-1} \simeq \mathcal{O}(1)$ is ample. We equip L with the canonical metric $2\phi_{FS}$. Then $H^0(X, L^k)$ may be identified with the space of all polynomials in $\mathbb{C}^n$ of degree at most $2k$, while the subspace $\mathcal{H}_{k,Z}$ is the space of polynomials of degree at most k. In this case the set D_Z (formula 4.1) is, by symmetry, a ball $B(0; r)$ centered at 0 of radius r, where r is determined by the following volume condition:

$$\int_{B(0;r)} (2\omega_{FS})^n/n! = 1.$$

Remark 5.3. The setting considered in [4] corresponds to replacing ω_{FS} by the Euclidean metric on $\mathbb{C}^n$ and the growth-condition 5.1 by the condition that

$$(1 + \epsilon) \ln(1 + |\zeta|^2) \leq \phi(\zeta) \tag{5.3}$$

(but with no upper bound or further assumption about smoothness “at infinity”). Hence, in a certain sense, example 5.1 may be seen as a limiting case of the setting in [4]. In particular, the set $D \cap \mathbb{C}^n$ may be non-compact in that example, while it is always compact under the assumption 5.3. Note however, that example 5.2 is essentially a special case of the situation studied in [4] (if ω_{FS} is replaced by the Euclidean metric). Indeed, the bound 5.3 holds with $\epsilon = 2$.

[3]in order that ϕ extend over the hyperplane at infinity to a *smooth* metric further conditions are needed.

In the next example, which in a certain sense is dual to the previous one, the line bundle L is not ample, but since $L \otimes [Z]^{-1}$ is, the main results in section 4 still apply.

Example 5.4. Let $X = \widetilde{\mathbb{P}^n}$ be the blow-up of $\mathbb{P}^n$ at the origin in $\mathbb{C}^n$ and denote by π the projection (blow-down map) from $\widetilde{\mathbb{P}^n}$ to $\mathbb{P}^n$. Let $L = \pi^*\mathcal{O}(2)$ and denote by Z the exceptional divisor over 0. Then $L \otimes [Z]^{-1}$ is ample (for example by the Nakai-Moishezon criteria [14]). We equip L with the metric $\pi^*(2\phi_{FS})$. Then $H^0(X, L^k)$ may again be identified with the space of all polynomials in $\mathbb{C}^n$ of total degree at most $2k$, while the subspace $\mathcal{H}_{k,Z}$ is the space of polynomials of total degree at least $k+1$. In this case the set D_Z (formula 4.1) is, by symmetry, the *complement* of a ball $B(0;r)$ of radius r, where r is determined by the following volume condition:

$$\int_{\mathbb{C}^n - B(0;r)} (2\omega_{FS})^n/n! = 1.$$

In order to compare with example 5.1, where there is no divisor, note that the blow-down map π induces an isomorphism $H^0(X, (L \otimes [Z]^{-1})^k) \simeq H^0(\mathbb{P}^n, \mathcal{O}(1)^k$ such that the "push-forward" of the corresponding singular metric ψ on $L \otimes [Z]^{-1}$ (compare the decomposition in lemma 4.1) becomes the singular metric

$$2\phi_{FS}(\zeta) - \ln|\zeta|^2$$

on $\mathcal{O}(1)$. This metric may be seen as a limit of metrics ϕ_{χ_i} on $\mathcal{O}(1)$ of the form 5.2, where the limiting function χ is given by

$$\chi(\zeta) = \ln(1 + |\zeta|^{-2}).$$

The point is that χ tends to 0 as $|\zeta|$ tends to infinity and to infinity as $|\zeta|$ tends to 0.

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